

ON THE MULTIFRACTAL ANALYSIS OF THE BRANCHING RANDOM WALK IN \mathbb{R}^d

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ABSTRACT. We establish the almost sure validity of the multifractal formalism for \mathbb{R}^d -valued branching random walks on the whole relative interior of the natural convex domain of study.

1. INTRODUCTION AND STATEMENT OF THE RESULT

This paper deals with the multifractal analysis of \mathbb{R}^d -valued branching random walks. The case $d = 1$ is now well known, but it turns out that extending the known results to higher dimensions is not a direct application of the method used in dimension 1. Let us start with the setting of the problem.

Let (N, X_1, X_2, \dots) be a random vector taking values in $\mathbb{N}_+ \times (\mathbb{R}^d)^{\mathbb{N}_+}$. Then consider $\{(N_u, X_{u1}, X_{u2}, \dots)\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$ be a family of independent copies of the vector (N, X_1, X_2, \dots) indexed by the set of finite words over the alphabet \mathbb{N}_+ (\mathbb{N}_+^0 contains the empty word denoted by \emptyset). Let T be the Galton-Watson tree with defining elements $\{N_u\}$: we have $\emptyset \in T$ and, if $u \in T$ and $i \in \mathbb{N}_+$ then ui , the concatenation of u and i , belongs to T if and only if $1 \leq i \leq N_u$. Similarly, for each $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, denote by $T(u)$ the Galton-Watson tree rooted at u and defined by the N_{uv} , $v \in \bigcup_{n \geq 0} \mathbb{N}_+^n$. For $n \geq 1$ and $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, denote $T(u) \cap \mathbb{N}_+^n$ by $T_n(u)$.

We assume that $\mathbb{E}(N) > 1$ so that the Galton-Watson tree is supercritical. Without loss of generality, we also assume that the probability of extinction is equal to 0, so that $\mathbb{P}(N \geq 1) = 1$.

For each infinite word $t = t_1 t_2 \dots \in \mathbb{N}_+^{\mathbb{N}_+}$ and $n \geq 0$, we set $t_{|n} = t_1 \dots t_n \in \mathbb{N}_+^n$. If $u \in \mathbb{N}_+^n$ for some $n \geq 0$, then n is the length of u and it is denoted by $|u|$ ($t_{|0} = \emptyset$). Then, we denote by $[u]$ the set of infinite words $t \in \mathbb{N}_+^{\mathbb{N}_+}$ such that $t_{|u} = u$.

The set $\mathbb{N}_+^{\mathbb{N}_+}$ is endowed with the standard ultrametric distance

$$d : (u, v) \mapsto e^{-\sup\{|w| : u \in [w], v \in [w]\}},$$

with the convention $\exp(-\infty) = 0$. The boundary of the Galton-Watson tree T is defined as the compact set

$$\partial T = \bigcap_{n \geq 1} \bigcup_{u \in T_n} [u],$$

consisting of the infinite words $t = t_1 t_2 \cdots$ over \mathbb{N}_+ such that for all $n \geq 0$, $t_{|n} = t_1 \cdots t_n \in T$.

After the strong law of large numbers, we know that, given $t \in \partial T$, we have, if the components of X are integrable and i.i.d., $\lim_{n \rightarrow \infty} \frac{1}{n} S_n(t) = \mathbb{E}(X)$ almost surely, where $S_n(t) = \sum_{k=1}^n X_{t_1 \cdots t_k}$. Since ∂T is not countable, the following question naturally arises : are there some $t \in \partial T$ so that $\lim_{n \rightarrow \infty} \frac{1}{n} S_n(t) = \alpha \neq \mathbb{E}(X)$? Multifractal analysis is a framework adapted to answer this question. Consider the set \mathcal{I} of those $\alpha \in \mathbb{R}^d$ such that

$$E(\alpha) = \left\{ t \in \partial T : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_{u_1 \cdots u_k} = \alpha \right\} \neq \emptyset.$$

These level sets can be described geometrically through their Hausdorff dimensions. They have been studied by many authors when $d = 1$, see for instance [11, 10, 15, 2, 6]; all these papers also deal with the multifractal analysis of associated Mandelbrot measures (see also [12, 17, 14] for the study of Mandelbrot measures dimension).

The vector space \mathbb{R}^d is endowed with the canonical scalar product and the associated euclidean norm respectively denoted $\langle \cdot | \cdot \rangle$ and $\| \cdot \|$. For all $x \in \mathbb{R}^d$ and $r \geq 0$, $B(x, r)$ stands for the closed Euclidean ball of radius r centered at x .

We will state our main result by using the notion of multifractal formalism (see [18] for an abstract vectorial multifractal formalism). Let us define the pressure like function

$$P(q) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{u \in T_n} \exp(\langle q | S_n(u) \rangle) \right) \quad (q \in \mathbb{R}^d).$$

Let P^* stand for the Legendre transform of the function P , where by convention the Legendre transform of a mapping $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is defined as the concave and upper semi-continuous function :

$$f^*(\alpha) := \inf_{q \in \mathbb{R}^d} \left(f(q) - \langle q, \alpha \rangle \right).$$

We say that the multifractal formalism holds at $\alpha \in \mathbb{R}^d$ if $\dim E(\alpha) = P^*(\alpha)$.

For the sake of simplicity we will assume throughout that the logarithmic moment generating function

$$\tilde{P}(q) : q \in \mathbb{R}^d \mapsto \log \mathbb{E} \left(\sum_{i=1}^N \exp (\langle q | X_i \rangle) \right),$$

is finite over \mathbb{R}^d (see Section 3 for the relaxation of this assumption).

Let

$$J = \left\{ q \in \mathbb{R}^d; \tilde{P}(q) - \langle q | \nabla \tilde{P}(q) \rangle > 0 \right\}.$$

Let

$$\Omega_\gamma^1 = \text{int} \left\{ q : \mathbb{E} \left[\left| \sum_{i=1}^N e^{\langle q | X_i \rangle} \right|^\gamma \right] < \infty \right\}, \quad \Omega^1 = \bigcup_{\gamma \in (1,2]} \Omega_\gamma^1,$$

and

$$\mathcal{J} = J \cap \Omega^1 \quad \text{and} \quad I = \left\{ \nabla \tilde{P}(q); q \in \mathcal{J} \right\}.$$

Our main result is the following.

Theorem 1.1. *Suppose that \tilde{P} is finite over \mathbb{R}^d . With probability 1, for all $\alpha \in I$, we have $\tilde{P}^*(\alpha) = P^*(\alpha)$ and the multifractal formalism holds at α , i.e., $\dim E(\alpha) = \tilde{P}^*(\alpha)$; in particular, $E(\alpha) \neq \emptyset$.*

In dimension 1, this result has been proved when N is not random in [2], and in the weaker form, for each fixed $\alpha \in I$, almost surely $\dim E(\alpha) = \tilde{P}^*(\alpha)$, when N is random in [11, 10, 15, 6]. Further comments on this result and its possible improvements are given in Section 3.

2. PROOF

2.1. Upper bounds for the Hausdorff dimension.

Proposition 2.1. *With probability 1, $P(q) \leq \tilde{P}(q)$ for all $q \in \mathbb{R}^d$, and then $P^*(\alpha) \leq \tilde{P}^*(\alpha)$, for all $\alpha \in \mathbb{R}^d$.*

Proof. The functions \tilde{P} and P being convex and thus continuous, we only need to prove the inequality $P(q) \leq \tilde{P}(q)$ for each $q \in \mathbb{R}^d$ almost surely. Fix $q \in \mathbb{R}^d$. For $s > \tilde{P}(q)$ we have

$$\begin{aligned} \mathbb{E} \left(\sum_{n \geq 1} e^{-ns} \sum_{u \in T_n} \exp (\langle q | S_n(u) \rangle) \right) &= \sum_{n \geq 1} e^{-ns} \mathbb{E} \left(\sum_{i=1}^N \exp (\langle q | X_i \rangle) \right)^n \\ &= \sum_{n \geq 1} e^{n(\tilde{P}(q)-s)}. \end{aligned}$$

Consequently, $\sum_{n \geq 1} e^{-ns} \sum_{u \in T_n} \exp(\langle q | S_n(u) \rangle) < \infty$ almost surely, so that we have $\sum_{u \in T_n} \exp(\langle q | S_n(u) \rangle) = O(e^{ns})$ and $P(q) \leq s$. Since $s > \tilde{P}(q)$ is arbitrary, we have the conclusion.

Proposition 2.2. *With probability 1, for all $\alpha \in \mathbb{R}^d$, $\dim E(\alpha) \leq P^*(\alpha)$, a negative dimension meaning that $E(\alpha)$ is empty.*

Proof. We have

$$\begin{aligned} E(\alpha) &= \bigcap_{\epsilon > 0} \bigcup_{N \in \mathbb{N}^*} \bigcap_{n \geq N} \left\{ t \in \partial T; \|S_n(t) - n\alpha\| \leq n\epsilon \right\} \\ &\subset \bigcap_{q \in \mathbb{R}^d} \bigcap_{\epsilon > 0} \bigcup_{N \in \mathbb{N}^*} \bigcap_{n \geq N} \left\{ t \in \partial T; |\langle q | S_n(t) - n\alpha \rangle| \leq n\|q\|\epsilon \right\}. \end{aligned}$$

Fix $q \in \mathbb{R}^d$ and $\epsilon > 0$. For $N \geq 1$, the set $E(q, N, \epsilon, \alpha) = \bigcap_{n \geq N} \{t \in \partial T; |\langle q | S_n(t) - n\alpha \rangle| \leq n\|q\|\epsilon\}$ is covered by the union of those $[u]$ such that $u \in T_n$, $n \geq N$, and $\langle q | S_n(u) - n\alpha \rangle + n\|q\|\epsilon \geq 0$.

We define the s -dimensional Hausdorff measure of a set E by

$$\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i \in \mathbb{N}} \text{diam}(U_i) \right\},$$

the infimum being taken over all the countable coverings $(U_i)_{i \in \mathbb{N}}$ of E of diameters less than or equal to δ .

Thus, for $s \geq 0$ and $n \geq N$,

$$\mathcal{H}_{e^{-n}}^s(E(q, N, \epsilon, \alpha)) \leq \sum_{u \in T_n} e^{-ns} \exp(\langle q | S_n(u) - n\alpha \rangle + n\|q\|\epsilon).$$

Consequently, if $\eta > 0$ and $s > P(q) + \eta - \langle q | \alpha \rangle + \|q\|\epsilon$, by definition of $P(q)$, for N large enough we have

$$\mathcal{H}_{e^{-n}}^s(E(q, N, \epsilon, \alpha)) \leq e^{-n\eta/2}.$$

This yield $\mathcal{H}^s(E(q, N, \epsilon, \alpha)) = 0$, hence $\dim E(q, N, \epsilon, \alpha) \leq s$. Since this holds for all $\eta > 0$ we get $\dim E(q, N, \epsilon, \alpha) \leq P(q) - \langle q | \alpha \rangle + \|q\|\epsilon$. It follows that

$$\dim E(\alpha) \leq \inf_{q \in \mathbb{R}^d} \inf_{\epsilon > 0} \sup_{N \in \mathbb{N}^*} P(q) - \langle q | \alpha \rangle + \|q\|\epsilon = P^*(\alpha).$$

If $P^*(\alpha) < 0$, we necessarily have $E(\alpha) = \emptyset$.

2.2. Lower bounds for the Hausdorff dimensions. For $(q, p) \in \mathcal{J} \times [1, \infty)$, we define the function

$$\phi(p, q) = e^{\tilde{P}(pq) - p\tilde{P}(q)}.$$

and for $q \in \mathcal{J}$ and $u \in T$, we define the sequence

$$Y_n(u, q) = \mathbb{E} \left(\sum_{i=1}^N e^{\langle q | X_i \rangle} \right)^{-n} \sum_{v \in T_n(u)} e^{\langle q | S_{|u|+n}(uv) - S_{|u|}(u) \rangle}, \quad (n \geq 1).$$

When $u = \emptyset$, $Y_n(\emptyset, q)$ will be denoted by $Y_n(q)$.

The sequence $(Y_n(u, q))_{n \geq 1}$ is a positive martingale with expectation 1, which converges almost surely and in L^1 norm to a positive random variable $Y(u, q)$ (see [12, 4] or [5, Theorem 1]). However, our study will need the almost sure simultaneous convergence of these martingales to positive limits (see Proposition 2.3(1)).

Let us state two propositions, the proof of which is postponed to the end of this section. The uniform convergence part of Proposition 2.3 is essentially Theorem 2 of [5], with slightly different assumptions. However, for the reader's convenience, and since the method used by Biggins will be used also in proving Propositions 2.4 and 2.7, we will include its proof. The second part of Proposition 2.3 defines the family of Mandelbrot measures built simultaneously to control the Hausdorff dimensions of the sets $E(\nabla P(q))$, $q \in \mathcal{J}$, from below. Then Proposition 2.4 introduces suitable logarithmic moment generating functions associated with these measures to get the desired lower bounds via large deviations inequalities.

Proposition 2.3. (1) *Let K be a compact subset of \mathcal{J} . There exists $p_K \in (1, 2]$ such that for all $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, the continuous functions $q \in K \mapsto Y_n(u, q)$ converge uniformly, almost surely and in L_{p_K} norm, to a limit $q \in K \mapsto Y(u, q)$. In particular, $\mathbb{E}(\sup_{q \in K} Y(u, q)^{p_K}) < \infty$. Moreover, $Y(u, \cdot)$ is positive almost surely.*

In addition, for all $n \geq 0$, $\sigma(\{(X_{u1}, \dots, X_{uN(u)}), u \in T_n\})$ and $\sigma(\{Y(u, \cdot), u \in T_{n+1}\})$ are independent, and the random functions $Y(u, \cdot), u \in T_{n+1}$, are independent copies of $Y(\cdot)$.

(2) *With probability 1, for all $q \in \mathcal{J}$, the weights*

$$\mu_q([u]) = \mathbb{E} \left(\sum_{i=1}^N e^{\langle q | X_i \rangle} \right)^{-|u|} e^{\langle q | S_{|u|}(u) \rangle} Y(u, q)$$

define a measure on ∂T .

For $q \in \mathcal{J}$, let

$$L_n(q, \lambda) = \frac{1}{n} \log \int_{\partial T} \exp(\langle \lambda | S_n(t) \rangle) d\mu_q(t), \quad (\lambda \in \mathbb{R}^d),$$

and

$$L(q, \lambda) = \limsup_{n \rightarrow \infty} L_n(q, \lambda).$$

Proposition 2.4. *Let K be a compact subset of \mathcal{J} . There exists a compact neighborhood Λ of the origin such that, with probability 1,*

$$(1) \quad \lim_{n \rightarrow \infty} \sup_{\lambda \in \Lambda} \sup_{q \in K} |L_n(q, \lambda) - (\tilde{P}(q + \lambda) - \tilde{P}(q))| = 0,$$

in particular $L(q, \lambda) = \tilde{P}(q + \lambda) - \tilde{P}(q)$ for $(q, \lambda) \in K \times \Lambda$.

Corollary 2.5. *With probability 1, for all $q \in \mathcal{J}$, for μ_q -almost every $t \in \partial T$,*

$$\lim_{n \rightarrow \infty} \frac{S_n(t)}{n} = \nabla \tilde{P}(q).$$

Proof. It follows from Proposition 2.4 that there exists $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$, and such that for all $\omega \in \Omega'$, for all $q \in \mathcal{J}$, there exists a neighborhood of 0 over which $L_n(q, \lambda)$ converges uniformly in λ towards $L(q, \lambda) = \tilde{P}(q + \lambda) - \tilde{P}(q)$.

For each $\omega \in \Omega'$, let us define for each $q \in \mathcal{J}$ the sequence of measures $\{\nu_{q,n}^\omega\}_{n \geq 1}$ as

$$(2) \quad \nu_{q,n}^\omega(B) = \mu_q(\{t \in \partial T : \frac{1}{n} S_n(t) \in B\})$$

for all Borel set $B \subset \mathbb{R}^d$. We denote $L(q, \lambda)$ by $L_q(\lambda)$. Since

$$L_n(q, \lambda) = \frac{1}{n} \log \int_{\mathbb{R}^d} \exp(n \langle \lambda | u \rangle) d\nu_{q,n}^\omega(u),$$

applying Gärtner-Ellis Theorem [8, Thm. 2.3.6], for all closed subsets Γ of \mathbb{R}^d , we have for all $q \in \mathcal{J}$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \nu_{q,n}^\omega(\Gamma) \leq \sup_{\alpha \in \Gamma} L_q^*(\alpha).$$

Let $\epsilon > 0$, and for each $q \in \mathcal{J}$ let $A_{q,\epsilon} = \{\alpha \in \mathbb{R}^d : d(\alpha, \nabla L_q(0)) \geq \epsilon\}$, where d is a Euclidean distance in \mathbb{R}^d . We have $\limsup_{n \rightarrow \infty} \frac{1}{n} \log \nu_{q,n}^\omega(A_{q,\epsilon}) \leq \sup_{\alpha \in A_{q,\epsilon}} L_q^*(\alpha)$. In addition, since $L_q(\lambda) = \tilde{P}(q + \lambda) - \tilde{P}(q)$ in a neighborhood of 0, we have $\nabla L_q(0) = \nabla \tilde{P}(q)$ and $L_q^*(\nabla L_q(0)) = 0 = \max L_q^*$. Moreover, since L_q is differentiable at 0, we have $L_q^*(\alpha) < L_q^*(\nabla L_q(0))$ for all $\alpha \neq \nabla L_q(0)$. Indeed, suppose that $L_q^*(\alpha) = 0$; then it follows from the definition of the Legendre transformation and the fact that $L_q(0) = 0$, that

$$\forall \lambda \in \mathbb{R}^d, \quad L_q(\lambda) \geq L_q(0) + \langle \lambda | \alpha \rangle,$$

hence α belongs to the subgradient of L_q at 0, which from Proposition 4.4 reduces to $\{\nabla L_q(0)\}$.

Now, due to the upper semi-continuity of the concave function L_q^* , we have $\gamma_{q,\epsilon} = \sup_{\alpha \in A_{q,\epsilon}} L_q^*(\alpha) < 0$.

Consequently, for all $q \in \mathcal{J}$, for n large enough, $\nu_{q,n}^\omega(A_{q,\epsilon}) \leq e^{n\gamma_{q,\epsilon}/2}$, i.e.

$$\mu_q(\{t \in \partial T : \frac{1}{n}S_n(t) \in A_{q,\epsilon}\}) \leq e^{n\gamma_{q,\epsilon}/2}.$$

Then it follows from the Borel-Cantelli Lemma (applied with respect to μ_q) that for all $q \in \mathcal{J}$, for μ_q -almost every $t \in \partial T$, we have $\frac{1}{n}S_n(t) \in B(\nabla \tilde{P}(q), \epsilon)$ for n large enough. Letting ϵ tend to 0 along a countable sequence yields the desired conclusion.

Corollary 2.6. *With probability 1, for all $q \in \mathcal{J}$, the sequence of random measure $(\nu_{q,n}^\omega)_{n \geq 1}$ defined in (2) satisfies the following large deviation property: for all λ in a neighborhood of 0,*

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \nu_{q,n}^\omega(B(\nabla L_q(\lambda), \epsilon)) = L_q^*(\nabla L_q(\lambda)),$$

where $L(q, \lambda) = \tilde{P}(q + \lambda) - \tilde{P}(q)$.

Proof. It is a consequence of Gärtner-Ellis theorem (see [8]).

We need a last proposition to get the lower bounds in Theorem 1.1. Its proof will end the section.

Proposition 2.7. *With probability 1, for all $q \in \mathcal{J}$, for μ_q -almost every $t \in \partial T$,*

$$\lim_{n \rightarrow \infty} \frac{\log Y(t_{|n}, q)}{n} = 0.$$

Proof of the lower bounds in Theorem 1.1: From Corollary 2.5, we have with probability 1, $\mu_q(E(\nabla \tilde{P}(q))) = 1$. In addition, with probability 1, for μ_q -almost every $t \in E(\nabla \tilde{P}(q))$, from the same corollary and Proposition 2.7, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log(\mu_q[t_{|n}])}{\log(\text{diam}([t_{|n}]))} &= \lim_{n \rightarrow \infty} \frac{-1}{n} \log \left(\exp(\langle q | S_n(t) \rangle - n\tilde{P}(q)) Y(t_{|n}, q) \right) \\ &= \tilde{P}(q) + \lim_{n \rightarrow \infty} \frac{\langle q | S_n(t) \rangle}{-n} - \lim_{n \rightarrow \infty} \frac{\log Y(t_{|n}, q)}{n} \\ &= \tilde{P}(q) - \langle q | \nabla \tilde{P}(q) \rangle = \tilde{P}^*(\nabla \tilde{P}(q)). \end{aligned}$$

We deduce the result from the mass distribution principle (Theorem 4.3).

Now, we give the proofs of the previous propositions.

2.3. Proofs of Propositions 2.3, 2.4 and 2.7. We start with several lemmas.

Lemma 2.8. *Recall that, for $(q, p) \in \mathcal{J} \times [1, \infty)$, $\phi(p, q) = e^{\tilde{P}(pq) - p\tilde{P}(q)}$. Then, for all nontrivial compact $K \subset \mathcal{J}$ there exists a real number $1 < p_K < 2$ such that for all $1 < p \leq p_K$ we have*

$$\sup_{q \in K} \phi(p_K, q) < 1.$$

Proof. Let $q \in \mathcal{J}$, one has $\frac{\partial \phi}{\partial p}(1^+, q) < 0$ and there exists $p_q > 1$ such that $\phi(p_q, q) < 1$. Therefore, in a neighborhood V_q of q , one has $\phi(p_q, q') < 1$ for all $q' \in V_q$. If K is a nontrivial compact of \mathcal{J} , it is covered by a finite number of such V_{q_i} . Let $p_K = \inf_i p_{q_i}$. If $1 < p \leq p_K$ and $\sup_{q \in K} \phi(p, q) \geq 1$, there exists $q \in K$ such that $\phi(p, q) \geq 1$, and $q \in V_{q_i}$ for some i . By log-convexity of the mapping $p \mapsto \phi(p, q)$ and the fact that $\phi(1, q) = 1$, since $1 < p \leq p_{q_i}$ we have $\phi(p, q) < 1$, which is a contradiction.

Lemma 2.9. *For all compact $K \subset \mathcal{J}$, there exists $\tilde{p}_K > 1$ such that,*

$$\sup_{q \in K} \mathbb{E} \left(\left(\sum_{i=1}^N e^{\langle q | X_i \rangle} \right)^{\tilde{p}_K} \right) < \infty.$$

Proof. Since K is compact and the family of open sets $J \cap \Omega_\gamma^1$ increases to \mathcal{J} as γ decreases to 1, there exists $\gamma \in (1, 2]$ such that $K \subset \Omega_\gamma^1$. Take $\tilde{p}_K = \gamma$. The conclusion comes from the fact that the function $q \mapsto \mathbb{E} \left(\left(\sum_{i=1}^N e^{\langle q | X_i \rangle} \right)^{\tilde{p}_K} \right)$ is convex over $\Omega_{\tilde{p}_K}^1$ so continuous.

The next lemma comes from [5].

Lemma 2.10. *If $\{X_i\}$ is a family of integrable and independent complex random variables with $\mathbb{E}(X_i) = 0$, then $\mathbb{E}|\sum X_i|^p \leq 2^p \sum \mathbb{E}|X_i|^p$ for $1 \leq p \leq 2$.*

Lemma 2.11. *Let (N, V_1, V_2, \dots) be a random vector taking values in $\mathbb{N}_+ \times \mathbb{C}^{\mathbb{N}_+}$ and such that $\sum_{i=1}^N V_i$ is integrable and $\mathbb{E}(\sum_{i=1}^N V_i) = 1$. Let M be an integrable complex random variable. Consider $\{(N_u, V_{u1}, V_{u2}, \dots)\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$ a sequence of independent copies of (N, V_1, \dots, V_N) and $\{M_u\}_{u \in \bigcup_{n \geq 0} \mathbb{N}_+^n}$ a sequence of copies of M such that for all $n \geq 1$, the random variables $M(u)$, $u \in \mathbb{N}_+^n$, are independent, and independent of $\{(N_u, V_{u1}, V_{u2}, \dots)\}_{u \in \bigcup_{k=0}^{n-1} \mathbb{N}_+^k}$. We define the sequence $(Z_n)_{n \geq 0}$ by $Z_0 = \mathbb{E}(M)$ and for $n \geq 1$*

$$Z_n = \sum_{u \in T_n} \left(\prod_{k=1}^n V_{u|_k} \right) M(u).$$

Let $p \in (1, 2]$. There exists a constant C_p depending on p only such that for all $n \geq 1$

$$\mathbb{E}(|Z_n - Z_{n-1}|^p) \leq C_p \mathbb{E}(|M|^p) \left(\mathbb{E} \left(\sum_{i=1}^N |V_i|^p \right) \right)^{n-1} \left(\mathbb{E} \left(\sum_{i=1}^N |V_i|^p \right) + \mathbb{E} \left(\left| \sum_{i=1}^N V_i \right|^p \right) + 1 \right).$$

Proof. The definition of the process Z_n gives immediately that

$$(3) \quad Z_n - Z_{n-1} = \sum_{u \in T_{n-1}} \prod_{k=1}^{n-1} V_{u|_k} \left(\sum_{i=1}^{N_u} V_{ui} M(ui) - M(u) \right).$$

For each $n \geq 1$ let $\mathcal{F}_n = \sigma\{(N_u, V_{u1}, \dots) : |u| \leq n-1\}$ and let \mathcal{F}_0 be the trivial sigma-field. The random variable $Z_n - Z_{n-1}$ is a weighted sum of independent and identically distributed random variables with zero mean, namely the random variables $\sum_{i=1}^{N_u} V_{ui}M(ui) - M(u)$, which are independent of \mathcal{F}_{n-1} . Applying the Lemma 2.10 with $X_u = \prod_{k=1}^{n-1} V_{u|k} \left(\sum_{i=1}^{N_u} V_{ui}M(ui) - M(u) \right)$, $u \in T_n$, conditionally on \mathcal{F}_{n-1} , and noticing that the weights $\prod_{k=1}^{n-1} V_{u|k}$, $u \in T_{n-1}$, are \mathcal{F}_{n-1} -measurable, we get

$$\begin{aligned} \mathbb{E}(|Z_n - Z_{n-1}|^p) &= \mathbb{E}\left(\mathbb{E}(|Z_n - Z_{n-1}|^p \mid \mathcal{F}_{n-1})\right) \\ &\leq \mathbb{E}\left(2^p \sum_{u \in T_{n-1}} \prod_{k=1}^{n-1} |V_{u|k}|^p \mathbb{E}\left|\sum_{i=1}^{N_u} V_{ui}M(ui) - M(u)\right|^p\right). \end{aligned}$$

It is easy to see that $\mathbb{E}\left(\sum_{u \in T_{n-1}} \prod_{k=1}^{n-1} |V_{u|k}|^p\right) = \prod_{k=1}^{n-1} \mathbb{E}\left(\sum_{i=1}^N |V_i|^p\right)$. Using the inequality

$$(4) \quad |x + y|^r \leq 2^{r-1}(|x|^r + |y|^r), \quad (r > 1),$$

we get

$$\mathbb{E}\left(\left|\sum_{i=1}^{N_u} V_{ui}M(ui) - M(u)\right|^p\right) \leq 2^{p-1} \mathbb{E}\left(\left|\sum_{i=1}^{N_u} V_{ui}M(ui)\right|^p + \mathbb{E}(|M|^p)\right).$$

Write $M(ui) = M(ui) - \mathbb{E}(M(ui)) + \mathbb{E}(M(ui))$. Then from the inequality (4), we get

$$\begin{aligned} \mathbb{E}\left(\left|\sum_{i=1}^{N_u} V_{ui}M(ui)\right|^p\right) &= \mathbb{E}\left(\left|\sum_{i=1}^{N_u} V_{ui}(M(ui) - \mathbb{E}(M(ui))) + V_{ui}\mathbb{E}(M(ui))\right|^p\right) \\ &\leq 2^{p-1} \mathbb{E}\left(\left|\sum_{i=1}^{N_u} V_{ui}(M(ui) - \mathbb{E}(M(ui)))\right|^p\right) + 2^{p-1} \mathbb{E}(|M|^p) \mathbb{E}\left(\left|\sum_{i=1}^{N_u} V_{ui}\right|^p\right). \end{aligned}$$

It follows from the Lemma 2.10 applied with $X_i = V_{ui}(M(ui) - \mathbb{E}(M(ui)))$ conditionally on $(N_u, V_{u1}, \dots, V_{uN_u})$, and from the independence of $M(ui)$ and $(N_u, V_{u1}, \dots, V_{uN_u})$, that

$$\begin{aligned} \mathbb{E}\left(\left|\sum_{i=1}^{N_u} V_{ui}(M(ui) - \mathbb{E}(M(ui)))\right|^p\right) &\leq 2^p \mathbb{E}\left(\sum_{i=1}^{N_u} |V_{ui}(M(ui) - \mathbb{E}(M(ui)))|^p\right) \\ &\leq 2^p \mathbb{E}(|M(u) - \mathbb{E}(M(u))|^p) \mathbb{E}\left(\sum_{i=1}^{N_u} |V_{ui}|^p\right) \\ &\leq 2^{2p} \mathbb{E}(|M|^p) \mathbb{E}\left(\sum_{i=1}^N |V_i|^p\right). \end{aligned}$$

Finally, we have

$$\mathbb{E}\left(\left|\sum_{i=1}^{N_u} V_{ui} M(ui) - M(u)\right|^p\right) \leq C_p \mathbb{E}|M|^p \left(\mathbb{E}\left(\sum_{i=1}^N |V_i|^p\right) + \mathbb{E}\left(\left|\sum_{i=1}^N V_i\right|^p\right) + 1\right).$$

Now we prove Propositions 2.3, 2.4 and 2.7.

Proof of the Proposition 2.3: (1) Recall that the uniform convergence result uses an argument developed in [5]. Fix a compact $K \subset \mathcal{J}$. By Lemma 2.9 we can fix a compact neighborhood K' of K and $\tilde{p}_{K'} > 1$ such that

$$\sup_{q \in K'} \mathbb{E}\left(\left(\sum_{i=1}^N e^{\langle q | X_i \rangle}\right)^{\tilde{p}_{K'}}\right) < \infty.$$

By Lemma 2.8, we can fix $1 < p_K \leq \min(2, \tilde{p}_{K'})$ such that $\sup_{q \in K} \phi(p_K, q) < 1$. Then for each $q \in K$, there exists a neighborhood $V_q \subset \mathbb{C}^d$ of q , whose projection to \mathbb{R}^d is contained in K' , and such that for all $u \in T$ and $z \in V_q$, the random variable

$$W_z(u) = \frac{e^{\langle z | X_u \rangle}}{\mathbb{E}\left(\sum_{i=1}^N e^{\langle z | X_i \rangle}\right)}$$

is well defined, and we have

$$\sup_{z \in V_q} \phi(p_K, z) < 1,$$

where for all $z, z' \in \mathbb{C}^d$ we set $\langle z | z' \rangle = \sum_{i=1}^d z_i \bar{z}'_i$, and

$$\phi(p_K, z) = \frac{\mathbb{E}\left(\sum_{i=1}^N |e^{\langle z | X_i \rangle}|^{p_K}\right)}{\left|\mathbb{E}\left(\sum_{i=1}^N e^{\langle z | X_i \rangle}\right)\right|^{p_K}}.$$

By extracting a finite covering of K from $\bigcup_{q \in K} V_q$, we find a neighborhood

$V \subset \mathbb{C}^d$ of K such that

$$\sup_{z \in V} \phi(p_K, z) < 1.$$

Since the projection of V to \mathbb{R}^d is included in K' and the mapping $z \mapsto \mathbb{E}\left(\sum_{i=1}^N e^{\langle z | X_i \rangle}\right)$ is continuous and does not vanish on V , by considering a smaller neighborhood of K included in V if necessary, we can assume that

$$A_V = \sup_{z \in V} \mathbb{E}\left(\left|\sum_{i=1}^N e^{\langle z | X_i \rangle}\right|^{p_K}\right) \left|\mathbb{E}\left(\sum_{i=1}^N e^{\langle z | X_i \rangle}\right)\right|^{-p_K} + 1 < \infty.$$

Now, for $u \in T$, we define the analytic extension to V of $Y_n(u, q)$ given by

$$\begin{aligned} Y_n(u, z) &= \sum_{v \in T_n(u)} W_z(u \cdot v_1) \cdots W_z(u \cdot v_1 \cdots v_n) \\ &= \mathbb{E} \left(\sum_{i=1}^N e^{\langle z | X_i \rangle} \right)^{-n} \sum_{v \in T_n(u)} e^{\langle z | S_{|u|+n} X(uv) - S_{|u|}(u) \rangle}. \end{aligned}$$

We denote also $Y_n(\emptyset, z)$ by $Y_n(z)$. Now, applying Lemma 2.11, with $V_i = e^{\langle z | X_i \rangle} / \mathbb{E} \left(\sum_{j=1}^N e^{\langle z | X_j \rangle} \right)$ and $M = 1$, we get

$$\begin{aligned} &\mathbb{E}(|Y_n(z) - Y_{n-1}(z)|^{p_K}) \\ &\leq C_{p_K} \left(\mathbb{E} \left(\sum_{i=1}^N |V_i|^{p_K} \right) \right)^{n-1} \left(\mathbb{E} \left(\sum_{i=1}^N |V_i|^{p_K} \right) + \mathbb{E} \left(\left| \sum_{i=1}^N V_i \right|^{p_K} \right) + 1 \right). \end{aligned}$$

Notice that $\mathbb{E} \left(\sum_{i=1}^N |V_i|^{p_K} \right) = \phi(p_K, z)$. Then,

$$\begin{aligned} &\mathbb{E}(|Y_n(z) - Y_{n-1}(z)|^{p_K}) \\ &\leq C_{p_K} \sup_{z \in V} \phi(p_K, z)^n + C_{p_K} A_V \sup_{z \in V} \phi(p_K, z)^{n-1}. \end{aligned}$$

With probability 1, the functions $z \in V \mapsto Y_n(z), n \geq 0$, are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset V$. Theorem (4.2) gives

$$\sup_{z \in D(z_0, \rho)} |Y_n(z) - Y_{n-1}(z)| \leq 2^d \int_{[0,1]^d} |Y_n(\zeta(\theta)) - Y_{n-1}(\zeta(\theta))| d\theta,$$

where, for $\theta = (\theta_1, \dots, \theta_d) \in [0, 1]^d$,

$$\zeta(\theta) = z_0 + 2\rho(e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_d}) \text{ and } d\theta = d\theta_1 \cdots d\theta_d.$$

Furthermore Jensen's inequality and Fubini's Theorem give

$$\begin{aligned} &\mathbb{E} \left(\sup_{z \in D(z_0, \rho)} |Y_n(z) - Y_{n-1}(z)|^{p_K} \right) \\ &\leq \mathbb{E} \left(\left(2^d \int_{[0,1]^d} |Y_n(\zeta(\theta)) - Y_{n-1}(\zeta(\theta))| d\theta \right)^{p_K} \right) \\ &\leq 2^{dp_K} \mathbb{E} \left(\int_{[0,1]^d} |Y_n(\zeta(\theta)) - Y_{n-1}(\zeta(\theta))|^{p_K} d\theta \right) \\ &\leq 2^{dp_K} \int_{[0,1]^d} \mathbb{E} |Y_n(\zeta(\theta)) - Y_{n-1}(\zeta(\theta))|^{p_K} d\theta \\ &\leq 2^{dp_K} C_{p_K} \sup_{z \in V} \phi(p_K, z)^n + C_{p_K} \sup_{z \in V} \phi(p_K, z)^{n-1} A_V. \end{aligned}$$

Since $\sup_{z \in V} \phi(p_K, z) < 1$, it follows that $\sum_{n \geq 1} \left\| \sup_{z \in D(z_0, \rho)} |Y_n(z) - Y_{n-1}(z)| \right\|_{p_K} < \infty$. This implies, $z \mapsto Y_n(z)$ converge uniformly, almost surely and in L^{p_K}

norm over the compact $D(z_0, \rho)$ to a limit $z \mapsto Y(z)$. This also implies that

$$\left\| \sup_{z \in P(z_0, \rho)} Y(z) \right\|_{p_K} < \infty.$$

Since K can be covered by finitely many such polydiscs $D(z_0, \rho)$ we get the uniform convergence, almost surely and in L^{p_K} norm, of the sequence $(q \in K \mapsto Y_n(q))_{n \geq 1}$ to $q \in K \mapsto Y(q)$. Moreover, since \mathcal{J} can be covered by a countable union of such compact K we get the simultaneous convergence for all $q \in \mathcal{J}$. The same holds simultaneously for all the function $q \in \mathcal{J} \mapsto Y_n(u, q)$, $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, because $\bigcup_{n \geq 0} \mathbb{N}_+^n$ is countable.

To finish the proof of Proposition 2.3(1), we must show that with probability 1, $q \in K \mapsto Y(q)$ does not vanish. Without loss of generality we can suppose that $K = [0, 1]^d$. If I is a dyadic closed subcube of $[0, 1]^d$, we denote by E_I the event $\{\exists q \in I : Y(q) = 0\}$. Let $I_0, I_1, \dots, I_{2^d-1}$ stand for the 2^d dyadic subcubes of I in the next generation. The event E_I being a tail event of probability 0 or 1, if we suppose that $P(E_I) = 1$, there exists $j \in \{0, 1, \dots, 2^d - 1\}$ such that $P(E_{I_j}) = 1$. Suppose now that $P(E_K) = 1$. The previous remark allows to construct a decreasing sequence $(I(n))_{n \geq 0}$ of dyadic subcubes of K such that $P(E_{I(n)}) = 1$. Let q_0 be the unique element of $\bigcap_{n \geq 0} I(n)$. Since $q \mapsto Y(q)$ is continuous we have $P(Y(q_0) = 0) = 1$, which contradicts the fact that $(Y_n(q_0))_{n \geq 1}$ converge to $Y(q_0)$ in L^1 .

(2) It is a consequence of the branching property

$$Y_{n+1}(u, q) = \sum_{i=1}^N e^{\langle q | X_{ui} \rangle - \tilde{P}(q)} Y_n(ui, q).$$

Proof of Proposition 2.4: Let K be a compact subset of \mathcal{J} . For all $q \in K$, there exists a compact neighborhood Λ of the origin such that $\{q + \lambda : q \in K, \lambda \in \Lambda\} \subset \mathcal{J}$. Let $R = \{q + \lambda : q \in K, \lambda \in \Lambda\}$. For $q \in K$ and $\lambda \in \Lambda$ we define

$$Z_n(q, \lambda) = \sum_{u \in T_n} e^{\langle q + \lambda | S_n(u) \rangle - n \tilde{P}(q + \lambda)} Y(u, q).$$

As in the proof of Proposition 2.3, we can find $p_R \in (1, 2]$ and a neighborhood $V \times V_\Lambda \subset \mathbb{C}^d \times \mathbb{C}^d$ of $K \times \Lambda$ such that the function

$$Z_n(z, z') = \left(\mathbb{E} \left(\sum_{i=1}^N e^{\langle z + z' | X_i \rangle} \right) \right)^{-n} \sum_{u \in T_n} e^{\langle z + z' | S_n(u) \rangle} Y(u, z),$$

are well defined on $V \times V_\Lambda$, and

$$\begin{cases} \sup_{z' \in V_\Lambda} \sup_{z \in V} \phi(p_R, z + z') < 1, \\ A_{V \times V_\Lambda} = \sup_{(z, z') \in V \times V_\Lambda} \mathbb{E} \left(\left| \sum_{i=1}^N e^{\langle z + z' | X_i \rangle} \right|^{p_R} \right) \left| \mathbb{E} \left(\sum_{i=1}^N e^{\langle z + z' | X_i \rangle} \right) \right|^{-p_R} + 1 < \infty. \end{cases}$$

Suppose that for each $(z_0, z'_0) \in V \times V_\Lambda$ and $\rho > 0$ such that $D(z_0, 2\rho) \times D(z'_0, 2\rho) \subset V \times V_\Lambda$ we have

$$(5) \quad \sum_{n \geq 1} \mathbb{E} \left(\sup_{(z, z') \in D(z_0, \rho) \times D(z'_0, \rho)} |Z_n(z, z') - Z_{n-1}(z, z')|^{p_R} \right) < \infty.$$

then, with probability 1, $(z, z') \mapsto Z_n(z, z')$ converges uniformly on $D(z_0, \rho) \times D(z'_0, \rho)$ to a limit $Z(z, z')$, whose restriction to $K \times \Lambda$ can be shown to be positive, in the same way as $Y(\cdot)$ was shown to be positive. Since $K \times \Lambda$ can be covered by finitely many polydiscs of the previous form $D(z_0, \rho) \times D(z'_0, \rho)$, we get the almost sure uniform convergence of $Z_n(q, \lambda)$ over $K \times \Lambda$ to $Z(q, \lambda) > 0$, hence the almost sure uniform convergence of $\frac{1}{n} \log(Z_n(q, \lambda))$ to 0 over $K \times \Lambda$. Then the conclusion comes from the fact that, for $(q, \lambda) \in K \times \Lambda$, one has

$$Z_n(q, \lambda) = \frac{\exp(nL_n(q, \lambda))}{\exp(n\tilde{P}(q + \lambda) - n\tilde{P}(q))},$$

indeed,

$$\begin{aligned} L_n(q, \lambda) &= \frac{1}{n} \log \int_{\partial T} \exp(\langle \lambda | S_n(t) \rangle) d\mu_q(t) \\ &= \frac{1}{n} \log \sum_{u \in T_n} \exp(\langle \lambda | S_n(u) \rangle) \mu_q([u]) \\ &= \frac{1}{n} \log \sum_{u \in T_n} \exp(\langle q + \lambda | S_n(u) \rangle - n\tilde{P}(q)) Y(u, q). \end{aligned}$$

Now we prove (5). Given $(z, z') \in V \times V_\Lambda$, applying Lemma 2.11 with $V_i = e^{\langle z+z' | X_i \rangle} / \mathbb{E}(\sum_{j=1}^N e^{\langle z+z' | X_j \rangle})$ and $M = Y(z)$ we get

$$\begin{aligned} &\mathbb{E}(|Z_n(z, z') - Z_{n-1}(z, z')|^{p_R}) \\ &\leq C_{p_R} \mathbb{E}(|Y(z)|^{p_R}) (\phi(p_R, z + z')^n + A_{V \times V_\Lambda} \phi(p_R, z + z')^{n-1}). \end{aligned}$$

For $\tilde{z} = (z, z') \in V \times V_\Lambda$ and $n \geq 1$ let $M_n(\tilde{z}) = Z_n(z, z') - Z_{n-1}(z, z')$. With probability 1 the functions $\tilde{z} \in V \times V_\Lambda \mapsto M_n(\tilde{z})$, $n \geq 1$, are analytic. Fix a closed polydisc $D(\tilde{z}_0, 2\rho) \subset V \times V_\Lambda$ with $\rho > 0$. Theorem (4.2) gives

$$\sup_{\tilde{z} \in D(\tilde{z}_0, \rho)} |M_n(\tilde{z})| \leq 2^{2d} \int_{[0,1]^{2d}} |M_n(\zeta(\theta))| d\theta,$$

where, for $\theta = (\theta_1, \dots, \theta_{2d}) \in [0, 1]^{2d}$,

$$\zeta(\theta) = \tilde{z}_0 + 2\rho(e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_{2d}}) \text{ and } d\theta = d\theta_1 \cdots d\theta_{2d}.$$

Furthermore Jensen's inequality and Fubini's Theorem give

$$\begin{aligned}
& \mathbb{E} \left(\sup_{\tilde{z} \in D(\tilde{z}_0, \rho)} |M_n(\tilde{z})|^{p_R} \right) \\
& \leq \mathbb{E} \left((2^{2d} \int_{[0,1]^{2d}} |M_n(\zeta(\theta))| d\theta)^{p_R} \right) \\
& \leq 2^{2dp_R} \mathbb{E} \left(\int_{[0,1]^{2d}} |M_n(\zeta(\theta))|^{p_R} d\theta \right) \\
& \leq 2^{2dp_R} \int_{[0,1]^{2d}} \mathbb{E} |M_n(\zeta(\theta))|^{p_R} d\theta \\
& \leq 2^{2dp_R} C_{p_R} \mathbb{E} \left(\sup_{z \in V} |Y(z)|^{p_R} \right) \\
& \quad \cdot \left(\sup_{(z,z') \in V \times V_\Lambda} \phi(p_R, z + z')^n + A_{V \times V_\Lambda} \sup_{(z,z') \in V \times V_\Lambda} \phi(p_R, z + z')^{n-1} \right).
\end{aligned}$$

Since $\sup_{(z,z') \in V \times V_\Lambda} \phi(p_R, z + z') < 1$, we obtain the conclusion (5).

Proof of the Proposition 2.7 Let K be a compact subset of \mathcal{J} . For $a > 1$, $q \in K$ and $n \geq 1$, we set

$$E_{n,a}^+ = \{t \in \partial T : Y(t|_n, q) > a^n\},$$

and

$$E_{n,a}^- = \{t \in \partial T : Y(t|_n, q) < a^{-n}\}.$$

It is sufficient to show that for $E \in \{E_{n,a}^+, E_{n,a}^-\}$,

$$(6) \quad \mathbb{E} \left(\sup_{q \in K} \sum_{n \geq 1} \mu_q(E) \right) < \infty.$$

Indeed, if this holds, then with probability 1, for each $q \in K$ and $E \in \{E_{n,a}^+, E_{n,a}^-\}$ $\sum_{n \geq 1} \mu_q(E) < \infty$, hence by the Borel-Cantelli lemma, for μ_q -almost every $t \in \partial T$, if n is big enough we have

$$-\log a \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log Y(t|_n, q) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log Y(t|_n, q) \leq \log a.$$

Letting a tend to 1 along a countable sequence yields the result.

Let us prove (6) for $E = E_{n,a}^+$ (the case $E = E_{n,a}^-$ is similar). At first we have,

$$\begin{aligned}
\sup_{q \in K} \mu_q(E_{n,a}^+) &= \sup_{q \in K} \sum_{u \in T_n} \mu_q([u]) \mathbf{1}_{\{Y(u,q) > a^n\}} \\
&= \sup_{q \in K} \sum_{u \in T_n} e^{\langle q | S_n(u) \rangle} e^{-n\tilde{P}(q)} Y(u, q) \mathbf{1}_{\{Y(u,q) > a^n\}} \\
&\leq \sup_{q \in K} \sum_{u \in T_n} e^{\langle q | S_n(u) \rangle} e^{-n\tilde{P}(q)} (Y(u, q))^{1+\nu} a^{-n\nu}, \\
&\leq \sup_{q \in K} \sum_{u \in T_n} e^{\langle q | S_n(u) \rangle} e^{-n\tilde{P}(q)} M(u)^{1+\nu} a^{-n\nu},
\end{aligned}$$

where $M(u) = \sup_{q \in K} Y(u, q)$ and $\nu > 0$ is an arbitrary parameter. For $q \in K$ and $\nu > 0$, we set $H_n(q, \nu) = \sum_{u \in T_n} e^{\langle q | S_n(u) \rangle} e^{-n\tilde{P}(q)} M(u)^{1+\nu} a^{-n\nu}$.

For $q \in K$, we have $\mathbb{E}\left(\sum_{i=1}^N e^{\langle q | X_i \rangle}\right) = e^{\tilde{P}(q)} \neq 0$. Then, there exists a neighborhood $U_K \subset \mathbb{C}^d$ of K such that $\mathbb{E}\left(\sum_{i=1}^N e^{\langle z | X_i \rangle}\right) \neq 0$ for all $z \in U_K$.

Lemma 2.12. *Fix $a > 1$. For $z \in U_K$ and $\nu > 0$, let*

$$H_n(z, \nu) = \mathbb{E}\left(\sum_{i=1}^N e^{\langle z | X_i \rangle}\right)^{-n} \sum_{u \in T_n} e^{\langle z | S_n(u) \rangle} M(u)^{1+\nu} a^{-n\nu}.$$

There exists a neighborhood $V \subset \mathbb{C}^d$ of K and a positive constant C_K such that, for all $z \in V$, for all integer $n \geq 1$,

$$(7) \quad \mathbb{E}(|H_n(z, p_K - 1)|) \leq C_K a^{-n(p_K - 1)/2},$$

where p_K provided by Proposition (2.3).

Proof. For $z \in U_K$ and $\nu > 0$, let

$$\tilde{H}_1(z, \nu) = |\mathbb{E}\left(\sum_{i=1}^N e^{\langle z | X_i \rangle}\right)|^{-1} \mathbb{E}\left(\sum_{i=1}^N |e^{\langle z | X_i \rangle}|\right) a^{-\nu}.$$

Let $q \in K$. Since $\mathbb{E}(\tilde{H}_1(q, \nu)) = a^{-\nu}$, there exists a neighborhood $V_q \subset U_K$ of q such that for all $z \in V_q$ we have $\mathbb{E}(|\tilde{H}_1(z, \nu)|) \leq a^{-\nu/2}$. By extracting a finite covering of K from $\bigcup_{q \in K} V_q$, we find a neighborhood $V \subset U_K$ of K such that $\mathbb{E}(|\tilde{H}_1(z, \nu)|) \leq a^{-\nu/2}$ for all $z \in V$. Therefore,

$$\begin{aligned} \mathbb{E}(|H_n(z, \nu)|) &= |\mathbb{E}\left(\sum_{i=1}^N e^{\langle z | X_i \rangle}\right)|^{-n} \mathbb{E}\left(|\sum_{u \in T_n} e^{\langle z | S_n X(u) \rangle} M(u)^{1+\nu}|\right) a^{-n\nu} \\ &\leq |\mathbb{E}\left(\sum_{i=1}^N e^{\langle z | X_i \rangle}\right)|^{-n} \mathbb{E}\left(\sum_{u \in T_n} |e^{\langle z | S_n X(u) \rangle}| M(u)^{1+\nu}\right) a^{-n\nu} \end{aligned}$$

By Proposition (2.3) there exists $p_K \in (1, 2]$ such that for all $u \in \bigcup_{n \geq 0} \mathbb{N}_+^n$, $\mathbb{E}(M(u)^{p_K}) = \mathbb{E}(M(\emptyset)^{p_K}) = C_K < \infty$. Take $\nu = p_K - 1$ in the last calculation, it follows, from the independence of $\sigma(\{(X_{u_1}, \dots, X_{u_{N(u)}}), u \in T_{n-1}\})$

and $\sigma(\{Y(u, \cdot), u \in T_n\})$ for all $n \geq 1$, that

$$\begin{aligned} \mathbb{E}\left(\left|H_n(z, p_K - 1)\right|\right) &\leq \left|\mathbb{E}\left(\sum_{i=1}^N e^{\langle z | X_i \rangle}\right)\right|^{-n} \mathbb{E}\left(\sum_{i=1}^N \left|e^{\langle z | X_i \rangle}\right|\right)^n C_K a^{-n(p_K - 1)} \\ &= C_K \mathbb{E}\left(\left|\tilde{H}_1(z, p_K - 1)\right|\right)^n \leq C_K a^{-n(p_K - 1)/2}, \end{aligned}$$

then the Lemma is now proved.

With probability 1, the functions $z \in V \mapsto H_n(z, \nu)$ are analytic. Fix a closed polydisc $D(z_0, 2\rho) \subset V$, $\rho > 0$ such that $D(z_0, 2\rho) \subset V$. Theorem (4.2) gives

$$\sup_{z \in D(z_0, \rho)} |H_n(z, p_K - 1)| \leq 2^d \int_{[0,1]^d} |H_n(\zeta(\theta), p_K - 1)| d\theta,$$

where, for $\theta = (\theta_1, \dots, \theta_d) \in [0, 1]^d$,

$$\zeta(\theta) = z_0 + 2\rho(e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_d}) \text{ and } d\theta = d\theta_1 \cdots d\theta_d.$$

Furthermore Fubini's Theorem gives

$$\begin{aligned} \mathbb{E}\left(\sup_{z \in D(z_0, \rho)} |H_n(z, p_K - 1)|\right) &\leq \mathbb{E}\left(2^d \int_{[0,1]^d} |H_n(\zeta(\theta), p_K - 1)| d\theta\right) \\ &\leq 2^d \int_{[0,1]^d} \mathbb{E} |H_n(\zeta(\theta), p_K - 1)| d\theta \\ &\leq 2^d C_K a^{-n(p_K - 1)/2}. \end{aligned}$$

Since $a > 1$ and $p_K - 1 > 0$, we get (6).

3. REMARKS

- (1) To estimate the dimension of the measure μ_q , we could have introduced, the logarithmic generating functions

$$\tilde{L}_n(q, s) = \frac{1}{n} \log \int_{\partial T} \mu_q(x|_n)^s d\mu_q(x), \quad (q \in J, s \in \mathbb{R}),$$

and studied their convergence in the same way as $L_n(q, s)$ was studied in Proposition 2.4. However, we would have had to find an analytic extension of the mapping $q \mapsto Y(q)^{1+s}$, almost surely in a deterministic neighborhood of any compact subset of \mathcal{J} in order to apply the technique using Cauchy formula. It turns out that the existence of such an extension is not clear, but assuming its existence, the same approach as in the proof of Corollary 2.5 would give the Hausdorff dimension of μ_q . If we only seek for a result valid for each $q \in \mathcal{J}$ almost surely, then it is not hard to get the almost sure uniform convergence of $s \mapsto \tilde{L}_n(q, s)$ in a compact neighborhood of 0 towards $s \mapsto \tilde{P}(q(1+s)) - (1+s)\tilde{P}(q)$, and the same approach as that of Corollary 2.5 yields the dimension of μ_q .

- (2) The method used in this paper is not a direct extension of that used in [2] for the case $d = 1$ on homogeneous trees. Indeed, in [2] the complex extension is used to build simultaneously the measures μ_q , but the proof that, uniformly in q , μ_q is carried by $E(P'(q))$ and has a Hausdorff dimension $P(q) - qP'(q)$ uses a real analysis method, which seems hard to extend in general when $d \geq 2$. Indeed, such an extension should use the injection of Sobolev spaces of the form $W^{1,p}(U)$ (U an open subset of \mathbb{R}^d) into a space of Hölder continuous functions [16, p. 28] to control the uniform convergence of series like $\sum_{n \geq 1} Z_n(q, \lambda)$ in the proof of Proposition 2.4; however, such an inclusion requires $p > d \geq 2$, so that we leave the range of orders of moments for which we have nice controls thanks to Lemma 2.10.
- (3) Our assumptions can be relaxed as follows. We could assume that \tilde{P} is finite over a neighborhood V of 0, consider $\mathcal{J}_V = \{q \in V : \tilde{P}(q) - \langle q | \nabla \tilde{P}(q) \rangle > 0\} \cap \Omega^1$, Then the same conclusions as in Theorem 1.1 hold with $I = \{\nabla \tilde{P}(q) : q \in \mathcal{J}_V\}$.
- (4) Suppose that \tilde{P} is finite over \mathbb{R}^d , and without loss of generality that it is strictly convex. Then I is open, and one can show that $\bar{I} = \{\alpha \in \mathbb{R}^d : \tilde{P}^*(\alpha) \geq 0\}$. Even if $J \subset \Omega^1$ so that we achieved the multifractal analysis on I , it remains the non trivial question of the Hausdorff dimension of $E(\alpha)$ for $\alpha \in \partial I$. This problem cannot be solved by the method used in this paper. In dimension 1, this boundary consists of two points, and the question has been partially solved in [2] and completely in [3] by building a suitable random measure (not of Mandelbrot type) on $E(\alpha)$. It would be easy to adapt the same method to show here that if $\alpha \in \partial I$ is of the form $\nabla \tilde{P}(q)$ with $P^*(\alpha) = 0$, or if $\alpha \in \partial I$ and there exists $q_0 \in \mathbb{R}^d$ such that $\alpha = \lim_{\lambda \rightarrow \infty} \nabla \tilde{P}(\lambda q_0)$, then we have $E(\alpha) \neq \emptyset$ and $\dim E(\alpha) = \tilde{P}^*(\alpha)$. In [1], a new approach unifying the cases $\alpha \in I$ and $\alpha \in \partial I$ is used to prove that almost surely, for all $\alpha \in \bar{I}$ we have $E(\alpha) \neq \emptyset$ and $\dim E(\alpha) = \tilde{P}^*(\alpha)$, without any reference to Ω^1 .
- (5) It is worth mentioning that a simple consequence of the proof of the previous result is the following large deviation property, which could also be deduced from [5]: With probability 1,

$$\forall \alpha \in I, \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \#\{u \in T_n : \|S_n(u) - n\alpha\| \leq n\epsilon\} = \tilde{P}^*(\alpha).$$

Indeed this property essentially follows from the fact that for all $\beta \in B(\alpha, \epsilon)$, $\{[u] : u \in T_n, \|S_n(u) - n\alpha\| \leq n\epsilon\}$ form, for n large enough, a sequence of coverings of diameter tending to 0 of a subset E of $E(\beta)$ with $\dim E = \dim E(\beta) = \tilde{P}^*(\beta)$. Hence $\liminf_{n \rightarrow \infty} \frac{1}{n} \log \#\{u \in T_n : \|S_n(u) - n\alpha\| \leq n\epsilon\} \geq \sup_{\beta \in B(\alpha, \epsilon)} \tilde{P}^*(\beta)$; the other inequality

$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \#\{u \in T_n : \|S_n(u) - n\alpha\| \leq n\epsilon\} \leq \sup_{\beta \in B(\alpha, \epsilon)} \tilde{P}^*(\beta)$
follows from Chernoff inequalities.

4. APPENDIX

4.1. Cauchy formula in several variables. Let us recall the Cauchy formula for holomorphic functions in several variables.

Definition 4.1. Let $d \geq 1$, a subset D of \mathbb{C}^d is an open polydisc if there exist open discs D_1, \dots, D_d of \mathbb{C} such that $D = D_1 \times \dots \times D_d$. If we denote by ζ_j the centre of D_j , then $\zeta = (\zeta_1, \dots, \zeta_d)$ is the centre of D and if r_j is the radius of D_j then $r = (r_1, \dots, r_d)$ is the multiradius of D . The set $\partial D = \partial D_1 \times \dots \times \partial D_d$ is the distinguished boundary of D . We denote by $D(\zeta, r)$ the polydisc with center ζ and radius r .

Let $D = D(\zeta, r)$ be a polydisc of \mathbb{C}^d and $g \in C(\partial D)$ a continuous function on ∂D . We define the integral of g on ∂D as

$$\int_{\partial D} g(\zeta) d\zeta_1 \dots d\zeta_d = (2i\pi)^d r_1 \dots r_d \int_{[0,1]^d} g(\zeta(\theta)) e^{i2\pi\theta_1} \dots e^{i2\pi\theta_d} d\theta_1 \dots d\theta_d,$$

where $\zeta(\theta) = (\zeta_1(\theta), \dots, \zeta_d(\theta))$ and $\zeta_j(\theta) = \zeta_j + r_j e^{i2\pi\theta_j}$ for $j = 1, \dots, d$.

Theorem 4.2. Let $D = D(a, r)$ be polydisc in \mathbb{C}^d with a multiradius whose components are positive, and f be a holomorphic function in a neighborhood of D . Then, for all $z \in P$

$$f(z) = \frac{1}{(2i\pi)^d} \int_{\partial D} \frac{f(\zeta) d\zeta_1 \dots d\zeta_d}{(\zeta_1 - z_1) \dots (\zeta_d - z_d)}.$$

It follows that

$$(8) \quad \sup_{z \in D(a, r/2)} |f(z)| \leq 2^d \int_{[0,1]^d} |f(\zeta(\theta))| d\theta_1 \dots d\theta_d$$

4.2. Mass distribution principle.

Theorem 4.3. [9, Theorem 4.2] Let ν be a positive and finite Borel probability measure on a compact metric space (X, d) . Assume that $M \subseteq X$ is a Borel set such that $\nu(M) > 0$ and

$$M \subseteq \left\{ t \in X, \liminf_{r \rightarrow 0^+} \frac{\log \nu(B(t, r))}{\log r} \geq \delta \right\}.$$

Then the Hausdorff dimension of M is bounded from below by δ .

4.3. Subgradient of convex function. Let $f : \mathbb{R}^d \rightarrow \bar{\mathbb{R}}$, and $x \in \mathbb{R}^d$. A vector $\xi \in \mathbb{R}^d$ is said to be subgradient of f at x if

$$\forall y \in \mathbb{R}^d, \quad f(y) \geq f(x) + \langle \xi | y - x \rangle.$$

The set of all subgradient of f at x is denoted by $\partial f(x)$.

Proposition 4.4. [19] If f is convex and differentiable at x , then $\partial f(x) = \{\nabla f(x)\}$.

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REFERENCES

- [1] N. Attia, J. Barral, Hausdorff and packing spectra, large deviations and free energy for branching random walks in \mathbb{R}^d .
- [2] J. Barral, Continuity of the multifractal spectrum of a statistically self-similar measure, *J. Theoretic. Probab.*, 13 (2000), 1027-1060.
- [3] J. Barral, X. Jin, Multifractal analysis of complex random cascades. *Commun. Math. Phys.*, 219 (2010), 129-168.
- [4] J.D. Biggins, Martingale convergence in the branching random walk. *J. Appl. Probab.* 14 (1977), 25-37
- [5] J.D. Biggins, Uniform convergence of martingales in the branching random walk, *Ann. Probab.*, 20 (1992), 137-151.
- [6] J.D. Biggins, B.M. Hambly and O.D. Jones, Multifractal spectra for random self-similar measures via branching processes, *Adv. Appl. Probab.*, **43** (2011), 1-39.
- [7] P. Billingsley, *Ergodic theory and information*. John Wiley and Sons, Inc., New York, London, Sydney (1965).
- [8] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics* (New York). Springer-Verlag, New York, second edition, 1998.
- [9] K.J. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*. Wiley, 2nd Edition, 2003.
- [10] K.J. Falconer, The multifractal spectrum of statistically self-similar measures. *J. Theor. Probability* 7 (3) (1994), 681-702.
- [11] R. Holley, E.C. Waymire, Multifractal dimensions and scaling exponents for strongly bounded random fractals. *Ann. Appl. Probab.* 2 (1992), 819-845.
- [12] J. P. Kahane, J. Peyrière, Sur certaines martingales de B. Mandelbrot. *Adv. Math.* 22 (1976), 131-145
- [13] Q. Liu, On generalized multiplicative cascades, *Stoch. Proc. Appl.*, 86 (2000), 263-286.
- [14] Q. Liu, A. Rouault, On two measures defined on the boundary of a branching tree. *Classical and modern branching processes* (Minneapolis, MN, 1994), IMA Vol. Math. Appl., vol. 84, Springer, New York (1997), pp. 187-201.
- [15] G.M. Molchan, Scaling exponents and multifractal dimensions for independent random cascades. *Commun. Math. Phys.* 179 (1996), 681-702.
- [16] J. Necas, *Introduction to the Theory of Nonlinear Elliptic Equations*, Teubner, Leipzig, 1983.
- [17] J. Peyrière, Calculs de dimensions de Hausdorff, *Duke Math. J.*, 44 (1977), 591-601.
- [18] J. Peyrière, A vectorial multifractal formalism. In: *Fractal Geometry and Applications* (Lapidus M L, van Frankenhuijsen M (eds.)). *Proc. Symp. Pure Math.* AMS, Providence, RI. 72, Part 2 (2004), 217-230.
- [19] R. T. Rockafellar, *Convex analysis*, Princeton Univ. Press, 1969. *Proceedings of Symposia in Pure Mathematics*, 72.2 (2004), 217-230.

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